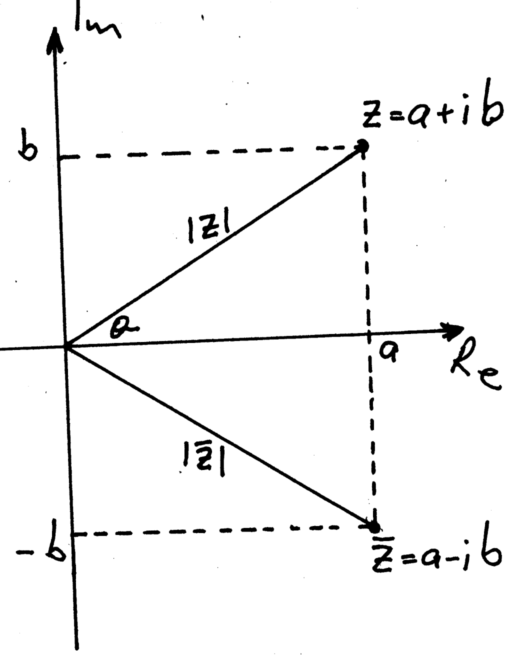
Kompleksni brojevi

θ - THETA
φ - FI

$3+i,\ 2,\ 4i,\ 7-5i,\ i$

$z=a+bi$ je kompleksan broj, $a,b\in\mathbb{R}$

Možemo ga predstaviti u kompleksnoj ravni:



$z\in\mathbb{C}$

$|z|=\sqrt{a^2+b^2}$ modul kompleksnog broja

$\bar{z}=a-ib$ konjugovano kompleksan broj

$\cos\theta=\dfrac{a}{|z|}$, $\sin\theta=\dfrac{b}{|z|}$, $tg\,\theta=\dfrac{b}{a}$

$i^2 \overset{def}{=} -1$

$i^4=i^2\cdot i^2=(-1)(-1)=1$, $i^{33}=(i^2)^{8}\cdot i=(-1)^2\cdot i=i$

$i^8=(i^2)^4=(-1)^4=1$, $i^{66}=(i^2)^{33}=(-1)^{33}=-1$, $i^{67}=(i^2)^{33}\cdot i=(-1)^{33}\cdot i=-i$

$z=|z|(\cos\theta+i\sin\theta)$ trigonometrijski oblik kompleksnog broja

$z=|z|e^{i\theta}$, $\theta\in[0,2\pi)$. Eulerov (eksponencijalni) oblik kompl. br.

$z_1=|z_1|(\cos\varphi_1+i\sin\varphi_1)$
$z_2=|z_2|(\cos\varphi_2+i\sin\varphi_2)$ $\Rightarrow$

$z_1=z_2$ akko $|z_1|=|z_2|$ i $(\varphi_1=\varphi_2+2k\pi)$

$z_1\cdot z_2=|z_1||z_2|\left[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\right]$

$\dfrac{z_1}{z_2}=\dfrac{|z_1|}{|z_2|}\left[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)\right]$, $z_2\neq 0$

$z=|z|(\cos\theta+i\sin\theta) \Rightarrow z^n=|z|^n\left[\cos(n\theta)+i\sin(n\theta)\right]$.

Teorema: Jednačina $z^n=w$, gdje je w po volji odabran kompleksan broj različit od nule ($0\in\mathbb{C}$), ima tačno n različitih rješenja:

$$z_k=\sqrt[n]{|w|}\left[\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right]$$

gdje je φ ($z=|z|(\cos\varphi+i\sin\varphi)$) najmanji pozitivan ugao iz $[0,2\pi)$ a $k=0,1,\dots,n-1$.

1. Zapisati u algebarskom obliku ($a+bi$, $a,b\in\mathbb{R}$) kompleksne brojeve a) $\frac{1}{1+i}$ b) $\frac{3+2i}{5-i}$.

Rj.

a) $\frac{1}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-i}{1-i^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$ $\quad Re\left(\frac{1}{1+i}\right)=\frac{1}{2}$ $\quad Im\left(\frac{1}{1+i}\right)=-\frac{1}{2}$

imaginarni dio broja

realni dio broja

b) $\frac{3+2i}{5-i}=\frac{3+2i}{5-i}\cdot\frac{5+i}{5+i}=\frac{15+3i+10i+2i^2}{25-i^2}=\frac{13+13i}{26}=\frac{1}{2}+\frac{1}{2}i$ $\quad Re\left(\frac{3+2i}{5-i}\right)=\frac{1}{2}=Im\left(\frac{3+2i}{5-i}\right)$

2. Odrediti kompleksan broj $z=a+bi$ koji zadovoljava jednačinu $|z|+z=2+i$.

Rj. $z=a+bi$

$|z|=\sqrt{a^2+b^2}$

$\sqrt{a^2+b^2}+a+bi=2+i \Rightarrow$

$$\begin{array}{l}\sqrt{a^2+b^2}+a=2\\ bi=i\\ \hline b=1\end{array}$$

$\sqrt{a^2+1}+a=2$

$\sqrt{a^2+1}=2-a \quad /^2$

$a^2+1=(2-a)^2$

$a^2+1=4-4a+a^2$

$4a=3$

$a=\frac{3}{4}$

Traženi kompleksan broj je $z=\frac{3}{4}+i$.

3. Odrediti skup tačaka (x,y) ravni koje zadovoljavaju jednačinu $yi+(5i-x^2)i+5=0$.

Rj. $yi+5i^2-x^2i+5=0 \Rightarrow yi-x^2i=0 \Rightarrow (y-x^2)i=0$

$\Rightarrow y-x^2=0 \Rightarrow y=x^2$ Traženi skup tačaka je parabola s jednačinom $y=x^2$.

4. Napisati kvadratnu jednačinu kojoj su $z_1=1+3i$ i $z_2=1-3i$ korijeni (rješenja).

Rj. $(x-x_1)(x-x_2)=0$

$(x-(1-3i))(x-(1+3i))=0$

$(x-1+3i)(x-1-3i)=0$

$x^2-x-3ix-x+1+\cancel{3i}+3ix-\cancel{3i}-9i^2=0$

$x^2-2x+10=0$

Kvadratna jednačina kojoj su z_1 i z_2 korijeni je $x^2-2x+10=0$.

5. Brojeve $z_1=-1+i$, $z_2=\sqrt{3}-i$, $z_3=-1-\sqrt{3}\,i$ predstaviti u trigonometriskom obliku, a zatim izračunati $\frac{z_1}{z_3}$, $z_1\cdot z_2$ i $(z_2)^{2010}$.

Rj. $z=a+ib=|z|(\cos\theta+i\sin\theta)$, $|z|=\sqrt{a^2+b^2}$, $\cos\theta=\frac{a}{|z|}$, $\sin\theta=\frac{b}{|z|}$

Prisjetimo se vrijednosti sin, cos, tg i ctg

	$30^\circ=\frac{\pi}{6}$ rad	$60^\circ=\frac{\pi}{3}$ rad	$45^\circ=\frac{\pi}{4}$ rad
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$tg\,\alpha$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1
$ctg\,\alpha$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

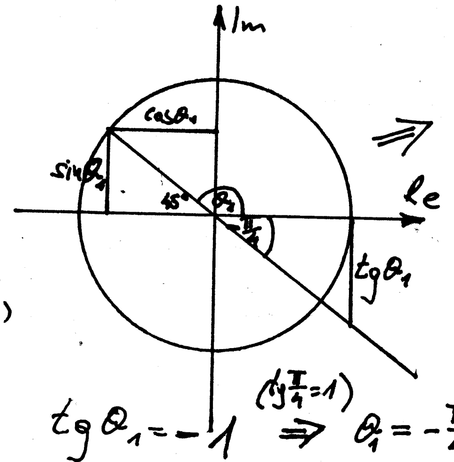
$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$z_1=-1+i$

$|z_1|=\sqrt{1+1}=\sqrt{2}$

$\cos\theta_1=\frac{-1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$,

$\sin\theta_1=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$, $tg\,\theta_1=-1 \overset{(tg\frac{\pi}{4}=1)}{\Rightarrow} \theta_1=-\frac{\pi}{4}$ ili $\theta_1=\frac{3\pi}{4}$



$\Rightarrow \theta_1=\frac{3\pi}{4}$

$z_2=\sqrt{3}-i$

$|z_2|=\sqrt{3+1}=2$

$\cos\theta_2=\frac{\sqrt{3}}{2}$

$\sin\theta_2=-\frac{1}{2}$

$tg\,\theta_2=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3} \overset{(tg\frac{\pi}{6}=\frac{\sqrt{3}}{3})}{\Rightarrow} \theta_2=-\frac{\pi}{6}$ ili $\theta_2=\frac{5\pi}{6}$

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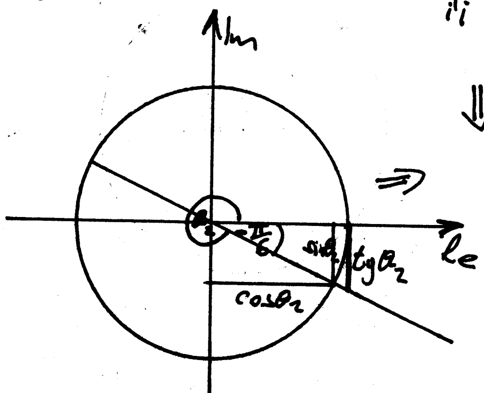


$\Rightarrow \theta_2=-\frac{\pi}{6}$

$z_3=-1-\sqrt{3}\,i$

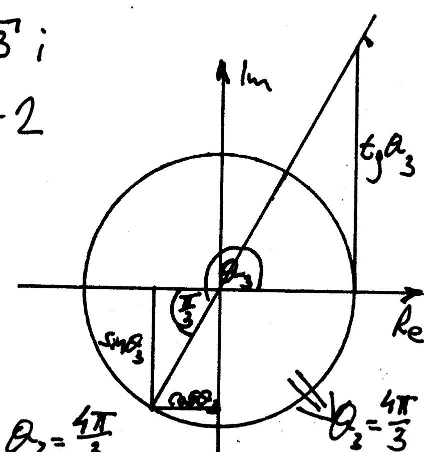
$|z_3|=\sqrt{1+3}=2$

$\cos\theta_3=-\frac{1}{2}$

$\sin\theta_3=-\frac{\sqrt{3}}{2}$

$tg\,\theta_3=\sqrt{3}\Rightarrow\theta_3=\frac{\pi}{3}$

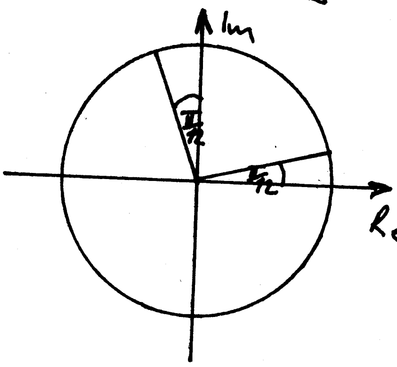
ili $\theta_3=\frac{4\pi}{3}$



$z_1=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$z_2=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

$z_3=2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$



$$\frac{z_1}{z_3}=\frac{\sqrt{2}}{2}\left(\cos\left(\frac{3\pi}{4}-\frac{4\pi}{3}\right)+i\sin\left(\frac{3\pi}{4}-\frac{4\pi}{3}\right)\right)$$

$$=\frac{\sqrt{2}}{2}\left(\cos\left(-\frac{7\pi}{12}\right)+i\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$=\frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right)=$$

$$=\frac{\sqrt{2}}{2}\left(-\sin\frac{\pi}{12}-i\cos\frac{\pi}{12}\right)=\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{6}-\sqrt{2}}{4}-i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$=\frac{-\sqrt{2}}{8}\left(\sqrt{6}-\sqrt{2}+i(\sqrt{6}+\sqrt{2})\right)$$

$$z_1\cdot z_2=2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}+\left(-\frac{\pi}{6}\right)\right)+i\sin\left(\frac{3\pi}{4}+\left(-\frac{\pi}{6}\right)\right)\right)=2\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=$$

$$=2\sqrt{2}\left(-\sin\frac{\pi}{12}+i\cos\frac{\pi}{12}\right)=2\sqrt{2}\left(-\frac{\sqrt{6}-\sqrt{2}}{4}+i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$z_2^{2010}=2^{2010}\left(\cos\left(2010\cdot\left(-\frac{\pi}{6}\right)\right)+i\sin\left(2010\cdot\left(-\frac{\pi}{6}\right)\right)\right)=$$

$$=2^{2010}\left(\cos(-335\pi)+i\sin(-335\pi)\right)=2^{2010}\left(\cos 335\pi-i\sin 335\pi\right)$$

$$=2^{2010}\left(\cos\pi-i\sin\pi\right)=2^{2010}(-1-0)=-2^{2010}.$$

6. Riješiti jednačinu $z^4=-4$ i rješenja predstaviti u kompleksnoj ravni.

Rj. Rješenja jednačine $z^4=-4$ su oblika

$$z_k=\sqrt[4]{|-4|}\left(\cos\frac{\varphi+2k\pi}{4}+i\sin\frac{\varphi+2k\pi}{4}\right),\quad k=0,1,2,3,\quad \varphi\in[0,2\pi)$$

$w=-4,\quad |w|=\sqrt{(-4)^2+0^2}=4,\quad \cos\varphi=\frac{-4}{4}=-1,\quad \sin\varphi=\frac{0}{4}=0\ \Rightarrow\ \varphi=\pi$ rad

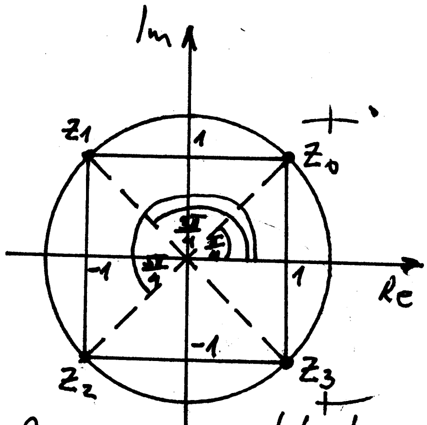
$$w=-4=4(\cos\pi+i\sin\pi)$$

$$z_0=\sqrt[4]{4}\left(\cos\frac{\pi+0}{4}+i\sin\frac{\pi+0}{4}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=1+i$$

$$z_1=\sqrt[4]{4}\left[\cos\frac{\pi+2\pi}{4}+i\sin\frac{\pi+2\pi}{4}\right]=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-1+i$$

$$z_2=\sqrt[4]{4}\left(\cos\frac{\pi+4\pi}{4}+i\sin\frac{\pi+4\pi}{4}\right)=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=-1-i$$

$$z_3=\sqrt[4]{4}\left(\cos\frac{\pi+6\pi}{4}+i\sin\frac{\pi+6\pi}{4}\right)=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=1-i$$



Rješenja predstavljena u kompleksnoj ravni

Rješenja jednačine $z^4=-4$ su $1+i$, $-1+i$, $-1-i$, $1-i$.

(7.) Izračunati $z=2^{-9}(b-2)^{18}$ ako je $b=3+2i-\frac{7-9i}{1-5i}$.

Rj. $b=3+2i-\frac{7-9i}{1-5i}=\frac{(3+2i)(1-5i)-(7-9i)}{1-5i}=\frac{3-15i+2i+10-7+9i}{1-5i}=\frac{6-4i}{1-5i}\cdot\frac{(1+5i)}{(1+5i)}=\frac{26+26i}{1+25}$

$b=1+i$, $(b-2)^2=(i-1)^2=-1-2i+1=-2i$, $(b-2)^{18}=\left[(b-2)^2\right]^9=(-2i)^9=-2^9\cdot i^9$

$z=2^{-9}(b-2)^{18}=2^{-9}\cdot(-2^9)\cdot i^8\cdot i=(-1)(i^2)^4\cdot i=-i$, $z=-i$ rješenje

(8.) Nađi sve vrijednosti korijena $\sqrt[4]{-2+2i\sqrt{3}}$.

Rj. $z=\sqrt[4]{w}$

$z^4=w$

$w=-2+2i\sqrt{3}$

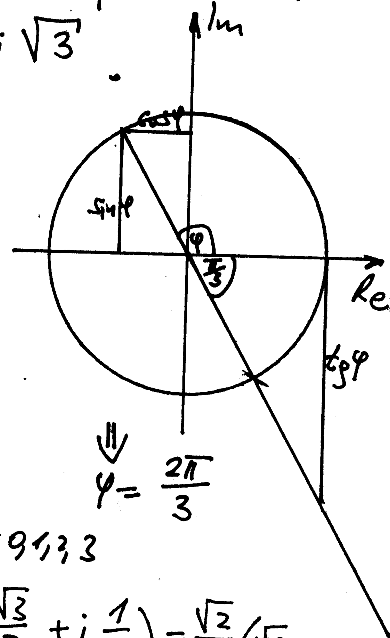
$|w|=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+12}=4$

$\cos\varphi=\frac{-2}{4}=-\frac{1}{2}$, $\sin\varphi=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$

$tg\,\varphi=\frac{2\sqrt{3}}{-2}=-\sqrt{3} \Rightarrow \varphi=-\frac{\pi}{3}$

$(tg\frac{\pi}{3}=\sqrt{2})$ ili $\varphi=\frac{2\pi}{3}$

$w=4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$



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$\varphi=\frac{2\pi}{3}$

Korijeni su oblika $z_k=\sqrt[4]{|w|}\left(\cos\frac{\varphi+2k\pi}{4}+i\sin\frac{\varphi+2k\pi}{4}\right)$, $k=0,1,2,3$

$z_0=\sqrt[4]{4}\left(\cos\frac{\frac{2\pi}{3}+0}{4}+i\sin\frac{\frac{2\pi}{3}+0}{4}\right)=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\sqrt{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\frac{\sqrt{2}}{2}(\sqrt{3}+i)$

$z_1=\sqrt[4]{4}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{4}+i\sin\frac{\frac{2\pi}{2}+2\pi}{4}\right)=\sqrt{2}\left(\cos\frac{8\pi}{12}+i\sin\frac{8\pi}{12}\right)=\sqrt{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$=\sqrt{2}\left(-\sin\frac{\pi}{6}+\cos\frac{\pi}{6}\right)=\sqrt{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{2}}{2}(-1+i\sqrt{3})$

$z_2=\sqrt{2}\left(\cos\frac{\frac{2\pi}{3}+4\pi}{4}+i\sin\frac{\frac{2\pi}{3}+4\pi}{4}\right)=\sqrt{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=\sqrt{2}\left(-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$

$=\sqrt{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\frac{\sqrt{2}}{2}(-\sqrt{3}-i)$

$z_3=\sqrt{2}\left(\cos\frac{\frac{2\pi}{3}+6\pi}{4}+i\sin\frac{\frac{2\pi}{3}+6\pi}{4}\right)=\sqrt{2}\left(\cos\frac{20\pi}{12}+i\sin\frac{20\pi}{12}\right)=\sqrt{2}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{2}}{2}(1-i\sqrt{3})$

(9.) ✓ Riješiti jednačinu $x^6+i=\sqrt{3}$.

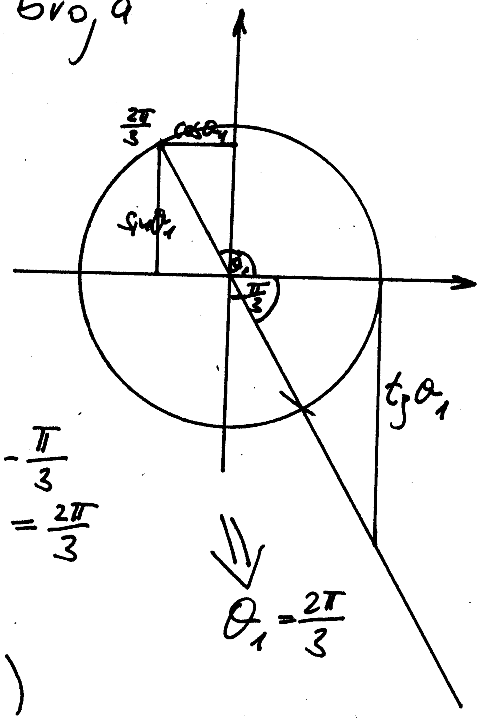
(10.) ✓ Izračunati $\left(\frac{1+i}{\sqrt{3}-i}\right)^5$.

(11.) ✓ Izračunati sve vrijednosti korijena $\sqrt[3]{i-1}$.

(12.) ✓ Nađi sve vrijednosti $\sqrt{z}$ (ima ih 4) ako je $z=(1+i)\sqrt{\sqrt{3}+i}$.

13. Odrediti realni i imaginarni dio broja

$$z=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{17}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$



Rj. $z=z_1^{17}\cdot z_2$

$z_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$|z_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

$\sin\theta_1=\frac{\frac{\sqrt{3}}{2}}{1}=\frac{\sqrt{3}}{2}$

$\cos\theta_1=-\frac{1}{2}$

$tg\,\theta_1=-\sqrt{3}\ \xRightarrow{tg\,60^\circ=\sqrt{3}}\ \theta_1=-\frac{\pi}{3}$ ili $\theta_1=\frac{2\pi}{3}$

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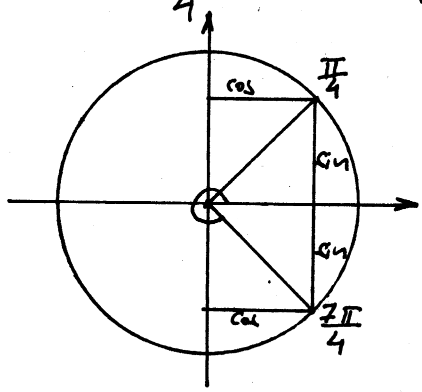
$\theta_1=\frac{2\pi}{3}$

$$z_1^{17}=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{17}=\cos 17\cdot\frac{2\pi}{3}+i\sin 17\cdot\frac{2\pi}{3}$$

$$z=z_1^{17}\cdot z_2=\left(\cos\frac{34\pi}{3}+i\sin\frac{34\pi}{3}\right)\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

$$=\cos\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)+i\sin\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)=\cos\frac{141\pi}{12}+i\sin\frac{141\pi}{12}$$

$$=\cos\frac{47\pi}{4}+i\sin\frac{47\pi}{4}=\cos 10\frac{7\pi}{4}+i\sin 10\frac{7\pi}{4}=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=$$

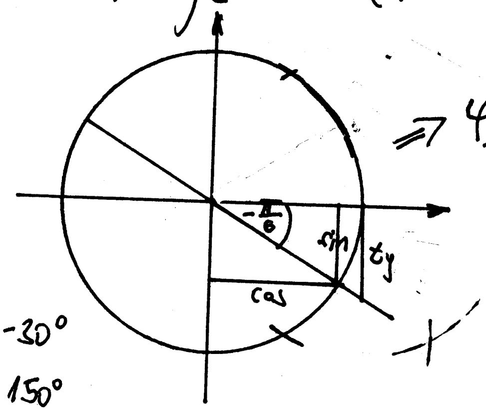


$$=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$Re(z)=\frac{\sqrt{2}}{2}$ — realni dio broja

$Im(z)=-\frac{\sqrt{2}}{2}$ — imaginarni dio broja

14. Naći sve vrijednosti korijena $\sqrt[3]{z}$ ako je $z=(\sqrt{3}-i)^9$.



Rj. $z=z_1^9$

$z_1=\sqrt{3}-i$

$|z_1|=\sqrt{3+1}=2$

$\cos\varphi_1=\frac{\sqrt{3}}{2}$

$\sin\varphi_1=-\frac{1}{2}$

$tg\,\varphi_1=-\frac{\sqrt{3}}{3}\ \xRightarrow{tg\,30^\circ=\frac{\sqrt{3}}{3}}\ \varphi_1=-30^\circ$ ili $\varphi_1=150^\circ$

$\Rightarrow \varphi_1=-\frac{\pi}{6}$

$z_1=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

$$z=z_1^9=2^9\left(\cos\left(-9\cdot\frac{\pi}{6}\right)+i\sin\left(-9\cdot\frac{\pi}{6}\right)\right)$$

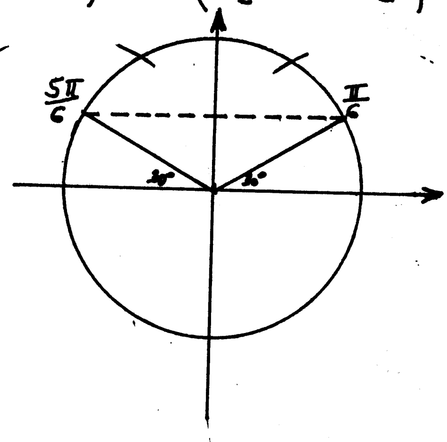
$$z=2^9\left(\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right)=2^9\left(\cos\frac{3\pi}{2}-i\sin\frac{3\pi}{2}\right)=2^9\cdot(-i)\cdot(-1)=2^9 i$$

$$z=2^9 i=2^9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$\sqrt[3]{z}$ računamo po formuli $z_k=\sqrt[3]{|z|}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3}\right)$

$$z_0 = \sqrt[3]{2^9}\left(\cos\frac{\frac{\pi}{2}}{3} + i\sin\frac{\frac{\pi}{2}}{3}\right) = \sqrt[3]{(2^3)^3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2^3\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

$$= 8\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 4(\sqrt{3}+i)$$



$$z_1 = 8\left(\cos\frac{\frac{\pi}{2}+2\pi}{3} + i\sin\frac{\frac{\pi}{2}+2\pi}{3}\right) = 8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= 8\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 8\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 4(-\sqrt{3}+i)$$

$$z_2 = 8\left(\cos\frac{\frac{\pi}{2}+4\pi}{3} + i\sin\frac{\frac{\pi}{2}+4\pi}{3}\right) = 8\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right)$$

$$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 8(0 + i(-1)) = -8i$$

Vrijednosti $\sqrt[3]{z}$ su $4(\sqrt{3}+i)$, $4(-\sqrt{3}+i)$ i $-8i$.

15. Nađi sve vrijednosti $\sqrt[3]{z}$ ako je $z = (\sqrt{3}-i)^5(1+i\sqrt{3})$.

Rj. $z = z_1^5 \cdot z_2 = (\sqrt{3}-i)^5 \cdot (1+i\sqrt{3}) = (\sqrt{3}-i)^4 \cdot (\sqrt{3}-i)(1+i\sqrt{3})$

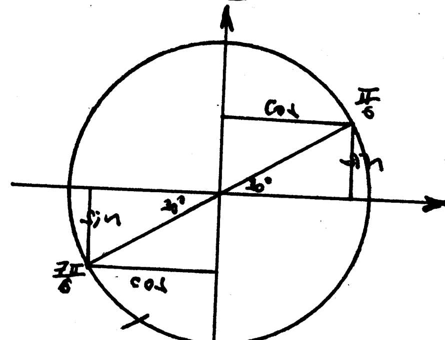
$$(\sqrt{3}-i)^2 = 3 - 2i\sqrt{3} + i^2 = 2 - 2\sqrt{3}$$

$$(\sqrt{3}-i)^4 = (2-2\sqrt{3})^2 = 4 - 8i\sqrt{3} + 4\cdot 3i^2 = -8 - 8i\sqrt{3}$$

$$(\sqrt{3}-i)(1+i\sqrt{3}) = \sqrt{3} + 3i - i - i^2\sqrt{3} = 2\sqrt{3} + 2i$$

$$z = (-8-8i\sqrt{3})(2\sqrt{3}+2i) = -16\sqrt{3} - 16i - 48i + 16\sqrt{3} = -64i = -2^6 i$$

$$z = 2^6\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$



$$z_k = \sqrt[3]{|z|}\left(\cos\frac{\varphi+2k\pi}{3} + i\sin\frac{\varphi+2k\pi}{3}\right)$$

$$z_0 = \sqrt[3]{2^6}\left(\cos\frac{-\frac{\pi}{2}}{3} + i\sin\frac{-\frac{\pi}{2}}{3}\right) = \sqrt[3]{(2^2)^3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 2(\sqrt{3}-i)$$

$$z_1 = 4\left(\cos\frac{-\frac{\pi}{2}+2\pi}{3} + i\sin\frac{-\frac{\pi}{2}+2\pi}{3}\right) = 4\left(\cos\frac{\frac{3\pi}{2}}{3} + i\sin\frac{\frac{3\pi}{2}}{3}\right) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 4(0 + i) = 4i$$

$$z_2 = 4\left(\cos\frac{-\frac{\pi}{2}+4\pi}{3} + i\sin\frac{-\frac{\pi}{2}+4\pi}{3}\right) = 4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 4\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

Tražena rješenja su $\sqrt[3]{z} \in \{2(\sqrt{3}-i),\ 4i,\ -2(\sqrt{3}+i)\}$

16. Riješiti jednačinu $\frac{(3+2i)(1+i)+2i}{(2-i)(1+i)-3} = \frac{7-i}{-4} \cdot z^4$.

Rj. $(3+2i)(1+i)+2i = 3+3i+2i+2i^2+2i = 1+7i$

$(2-i)(1+i)-3 = 2+2i-i-i^2-3 = i$

Sad jednačina $\frac{(3+2i)(1+i)+2i}{(2-i)(1+i)-3} = \frac{7-i}{-4} \cdot z^4$ postoje

$\frac{1+7i}{i} = \frac{7-i}{-4} \cdot z^4$, kako je $\frac{1+7i}{i} \cdot \frac{i}{i} = \frac{i+7i^2}{i^2} = \frac{-7+i}{-1} = 7-i$

imamo $7-i = \frac{7-i}{-4} \cdot z^4 \quad / \cdot \frac{1}{7-i}$

$1 = \frac{1}{-4} \cdot z^4 \Rightarrow z^4 = -4$ primetite da smo ovu jednačinu riješili u zadatku broj 6.

17. Napisati u trigonometriskom i eksponencijalnom obliku kompleksni broj $z = 2\sqrt{3}+2i$, a zatim naći $\sqrt[4]{z}$.

18. Napisati u trigonometriskom i eksponencijalnom obliku kompleksni broj $z = \frac{-1-i}{2}$, a zatim naći z^{14}.

19. Izračunati $z = 2^{-6}(a-2i)^{18}$, ako je $a = \frac{8+i}{3+2i} - 3 + 2i$.

20. Izračunati broj $z = \frac{\left(\frac{1}{2\sqrt{3}} - \frac{i}{2}\right)^9}{\left(-1+\frac{i}{\sqrt{3}}\right)^6}$.

21. Izračunati $\left(\frac{1+i\sqrt{3}}{2}\right)^{60} + \left(\frac{1-i\sqrt{3}}{2}\right)^{30}$.

22. Odrediti prirodan broj x iz uslova $(3+4i)^{x-1} - (1+i)^4 = 5^x$.

Napisati sva rješenja jednačine $x^4+x^2+1=0$ u trigonometriskom obliku.

Rj uvodimo smjenu $x^2=t$

$$t^2+t+1=0$$

$$D=1-4=-3=3i^2$$

$$t_{1,2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$t_1=\frac{-1-i\sqrt{3}}{2}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$t_2=\frac{-1+i\sqrt{3}}{2}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$t_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$t_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

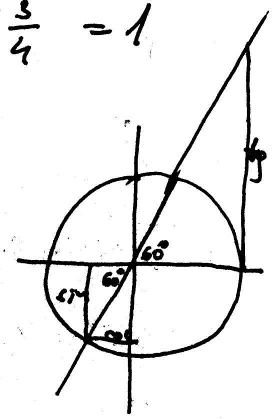
$$|t_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

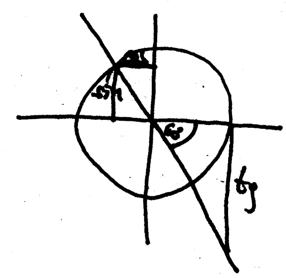
$$\cos\varphi_1=-\frac{1}{2}$$

$$\sin\varphi_1=-\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_1=\sqrt{3}$$

$$tg\,60^\circ=\sqrt{3}$$





$$\varphi_1=240^\circ=\frac{4\pi}{3}$$

$$|t_2|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\cos\varphi_2=-\frac{1}{2}$$

$$\sin\varphi_2=\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_2=-\sqrt{3}$$

$$\varphi_2=120^\circ=\frac{2\pi}{3}$$

$$x^2=t$$

$$z=\sqrt{t_1},\quad z_k=\sqrt{|t_1|}\left(\cos\frac{\varphi+2k\pi}{2}+i\sin\frac{\varphi+2k\pi}{2}\right),\quad k=0,1$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}}{2}+i\sin\frac{\frac{4\pi}{3}}{2}\right)=\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}+2\pi}{2}+i\sin\frac{\frac{4\pi}{3}+2\pi}{2}\right)=\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$$

$$z=\sqrt{t_2}$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}}{2}+i\sin\frac{\frac{2\pi}{3}}{2}\right)=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{2}+i\sin\frac{\frac{2\pi}{3}+2\pi}{2}\right)=\cos\frac{8\pi}{6}+i\sin\frac{8\pi}{6}=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$

Sva rješenja jednačine $x^4+x^2+1=0$ napisana u trigonometriskom obliku su:

$x_1=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $x_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $x_3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

i $x_4=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

Riješiti jednačinu $x^4+\frac{9}{4}=0$ i rješenja predstaviti u kompleksnoj ravni.

Rj. $x^4=-\frac{9}{4}$

$x=\sqrt[4]{-\frac{9}{4}}$

$x=\sqrt[4]{z}$

n-ti korijen kompleksnog broja tražimo po formuli:

$$z_k=\sqrt[n]{|z|}\left(\cos\frac{\omega+2k\pi}{n}+i\sin\frac{\omega+2k\pi}{n}\right),\quad k=1,2,\dots,n$$

$z=-\frac{9}{4}$ $\qquad |z|=\sqrt{\left(\frac{9}{4}\right)^2+0^2}=\frac{9}{4}$ $\qquad z=\frac{9}{4}(\cos\pi+i\sin\pi)$

$$\cos\omega=\frac{a}{|z|}=\frac{-\frac{9}{4}}{\frac{9}{4}}=-1$$

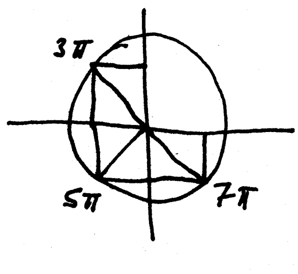
$$\sin\omega=\frac{0}{|z|}=0 \qquad \Longrightarrow \omega=\pi$$

$$z_0=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt[4]{\left(\frac{3}{2}\right)^2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$$

$$z_0=\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

$$z_1=\sqrt{\frac{3}{2}}\left(\cos\frac{\pi+2\pi}{4}+i\sin\frac{\pi+2\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

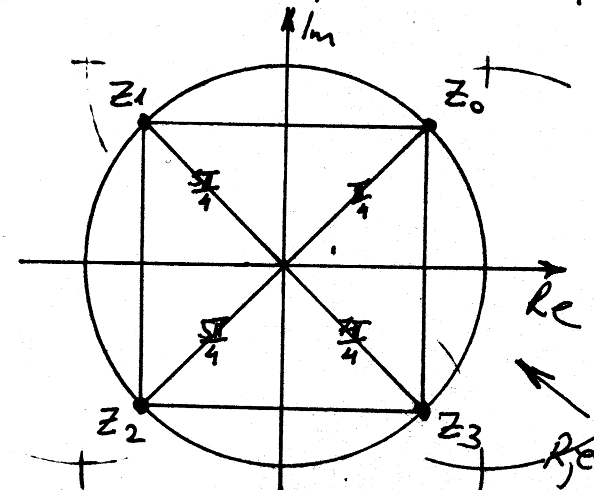


$$z_2=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+4\pi}{4}+i\sin\frac{5\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$

$$z_3=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+6\pi}{4}+i\sin\frac{7\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$



Rješenja jednačine su:

$\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

i $\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

Rješenja predstavljena u kompleksnoj ravni.

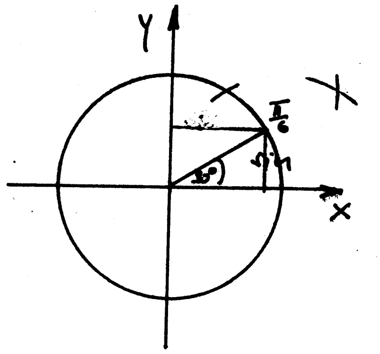
Izračunati $\dfrac{(\sqrt{3}+i)^{22}(1-i)^{15}}{(-1-i)^{3}}$.

R.j. $z_1=\sqrt{3}+i$

$|z_1|=\sqrt{3+1}=\sqrt{4}=2$

$\cos\theta_1=\dfrac{a}{|z_1|}=\dfrac{\sqrt{3}}{2}$

$\sin\theta_1=\dfrac{b}{|z_1|}=\dfrac{1}{2}$



$\sin 30°=\sin\frac{\pi}{6}=\frac{1}{2}$

$\theta_1=\dfrac{\pi}{6}$

$z_1=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$z_1^{22}=2^{22}\left(\cos\left(22\cdot\frac{\pi}{6}\right)+i\sin\left(22\cdot\frac{\pi}{6}\right)\right)$

$=2^{22}\left(\cos\frac{11\pi}{3}+i\sin\frac{11\pi}{3}\right)$

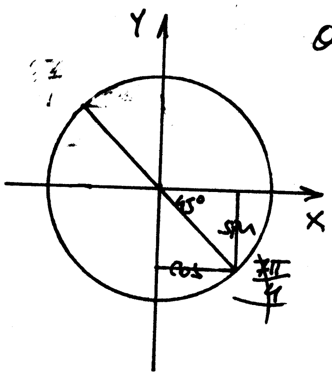
$z_2=1-i$

$|z_2|=\sqrt{1+1}=\sqrt{2}$

$\cos\theta_2=\dfrac{a}{|z_2|}=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$

$\sin\theta_2=\dfrac{b}{|z_2|}=\dfrac{-1}{\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$

$tg\,\theta_2=\dfrac{b}{a}=-1$



$tg\,45°=tg\frac{\pi}{4}=1$

$\theta_2=\dfrac{7\pi}{4}$

$z_2=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$z_2^{15}=(\sqrt{2})^{15}\left(\cos 15\cdot\frac{7\pi}{4}+i\sin 15\cdot\frac{7\pi}{4}\right)$

$=2^{7}\sqrt{2}\left(\cos\frac{105\pi}{4}+i\sin\frac{105\pi}{4}\right)$

$(1-i)^2=1-2i+i^2=-2i$

$(1-i)^{14}=\left((1-i)^2\right)^7=(-2i)^7=-2^7\cdot i^7=-2^7\cdot i^6\cdot i$

$=-2^7\cdot(i^2)^3\cdot i=2^7\cdot i$

$z_3=-1-i$

$(-1-i)^2=1+2i+i^2=1+2i-1=2i$

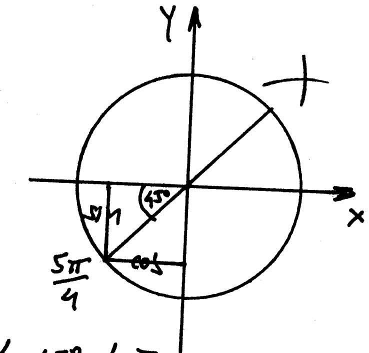
$(-1-i)^3=(-1-i)^2\cdot(-1-i)=2i(-1-i)=-2i-2i^2=2-2i$

$|z_3|=\sqrt{1+1}=\sqrt{2}$

$\cos\theta_3=\dfrac{a}{|z_3|}=\dfrac{-1}{\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$

$\sin\theta_3=\dfrac{b}{|z_3|}=\dfrac{-1}{\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$

$tg\,\theta_3=\dfrac{b}{a}=\dfrac{-1}{-1}=1$



$tg\,45°=tg\frac{\pi}{4}=1$

$z_3=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$z_3^{3}=(\sqrt{2})^{3}\left(\cos 3\cdot\frac{5\pi}{4}+i\sin 3\cdot\frac{5\pi}{4}\right)$

$=2\sqrt{2}\left(\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right)$

$\dfrac{(1-i)^{15}}{(-1-i)^3}=\dfrac{2^7\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}\left(\cos\dfrac{105\pi-15\pi}{4}+i\sin\dfrac{105-15}{4}\pi\right)=2^6\left(\cos\dfrac{90\pi}{4}+i\sin\dfrac{90\pi}{4}\right)$

$z_1^{22}\cdot\dfrac{z_2^{15}}{z_3^{3}}=2^{22}\cdot 2^{6}\left(\cos\left(\dfrac{11\pi}{3}+\dfrac{90\pi}{4}\right)+i\sin\left(\dfrac{11\pi}{3}+\dfrac{90\pi}{4}\right)\right)=2^{28}\left(\cos\dfrac{314}{12}\pi+i\sin\dfrac{157}{6}\pi\right)$

$z=2^{28}\left(\cos\left(\frac{\pi}{6}+2\cdot 13\pi\right)+i\sin\left(\frac{\pi}{6}+2\cdot 13\pi\right)\right)=2^{28}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2^{28}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=2^{27}(\sqrt{3}+i)$

(#) Nađi sve vrijednosti korijena $\sqrt[6]{-27}$.

Rj. Označimo sa $z=\sqrt[6]{-27}$

$$z^6=-27$$

Teorema Jednačina $z^n=w$, gdje je w po volji odabran kompleksan broj različit od 0 ima tačno n različitih rješenja koji su oblika

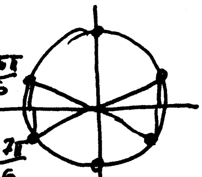
$$z_k=\sqrt[n]{|w|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$$

gdje je φ najmanji pozitivan ugao iz intervala $[0,2\pi)$ takav da $w=|w|(\cos\varphi+i\sin\varphi)$, a $k=0,1,2,\dots,n-1$.

U našem slučaju $w=-27 \Rightarrow |w|=\sqrt{(-27)^2+0^2}=27$

$w=a+ib$ $\quad$ $\left(|w|=\sqrt{a^2+b^2}\right)$

$$\left.\begin{aligned}\cos\varphi&=\frac{-27}{27}\left(=\frac{a}{|w|}\right)=-1\\ \sin\varphi&=\frac{b}{|w|}=\frac{0}{27}=0\end{aligned}\right\}\Rightarrow \varphi=\pi$$



$$w=-27=27(\cos\pi+i\sin\pi)$$

$$z_0=\sqrt[6]{27}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=(3^3)^{\frac{1}{6}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\sqrt{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$z_1=\sqrt[6]{27}\left(\cos\frac{\pi+2\pi}{6}+i\sin\frac{\pi+2\pi}{6}\right)=\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i\sqrt{3}$$

$$z_2=\sqrt[6]{27}\left(\cos\frac{\pi+4\pi}{6}+i\sin\frac{\pi+4\pi}{6}\right)=\sqrt{3}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=\sqrt{3}\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$z_3=\sqrt[6]{27}\left(\cos\frac{\pi+6\pi}{6}+i\sin\frac{\pi+6\pi}{6}\right)=\sqrt{3}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=\sqrt{3}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$$

$$z_4=\sqrt[6]{27}\left(\cos\frac{\pi+8\pi}{6}+i\sin\frac{\pi+8\pi}{6}\right)=\sqrt{3}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i\sqrt{3}$$

$$z_5=\sqrt[6]{27}\left(\cos\frac{\pi+10\pi}{6}+i\sin\frac{11\pi}{6}\right)=\sqrt{3}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\sqrt{3}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$$

Sve vrijednosti korijena $\sqrt[6]{-27}$ su: $\frac{3}{2}+i\frac{\sqrt{3}}{2}$, $i\sqrt{3}$, $-\frac{3}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{3}{2}-i\frac{\sqrt{3}}{2}$, $-i\sqrt{3}$ i $\frac{3}{2}-i\frac{\sqrt{3}}{2}$.

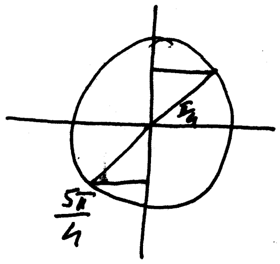
(#) Ako je $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, izračunati sve vrijednosti korijena $\sqrt[3]{\left(z+\frac{1}{z}+i\right)^5}$.

R:

$z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $\quad z+\frac{1}{z}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{1}{\frac{1-i\sqrt{3}}{2}}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2}{1-i\sqrt{3}}\cdot\frac{(1+i\sqrt{3})}{(1+i\sqrt{3})}$

$=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2+2i\sqrt{3}}{1+3}=\frac{1}{2}-\cancel{i\frac{\sqrt{3}}{2}}+\frac{1}{2}+\cancel{i\frac{\sqrt{3}}{2}}=1$

$z+\frac{1}{z}+i=1+i$ $\quad$ Uvedimo oznaku $w=z+\frac{1}{z}+i=1+i$



$|w|=\sqrt{2}$

$$\left.\begin{aligned}\cos\varphi&=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \sin\varphi&=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ tg\,\varphi&=1\end{aligned}\right\}\Rightarrow \varphi=45^\circ=\frac{\pi}{4}\text{ rad}$$

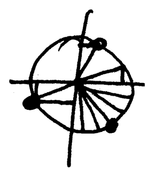
$w=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$w^5=(\sqrt{2})^5\left(\cos 5\cdot\frac{\pi}{4}+i\sin 5\cdot\frac{\pi}{4}\right)=4\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$w^n=c$ gdje je c kompleksan broj ima tačno n rješenje

$w_k=\sqrt[n]{|c|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$, φ najmanji pozitivan ugao iz $[0,2\pi)$
$k=1,2,\dots,n$

Mi treba da nađemo $\sqrt[3]{\left(z+\frac{1}{z}+i\right)^5}$ tj. $\sqrt[3]{w^5}$



$v_1=\sqrt[3]{4\sqrt{2}}\left(\cos\frac{\frac{5\pi}{4}+0}{3}+i\sin\frac{5\pi}{12}\right)=32^{\frac{1}{6}}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)=$

$=\sqrt[6]{32}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$

$v_2=\sqrt[6]{32}\left(\cos\frac{\frac{5\pi}{4}+2\pi}{3}+i\sin\frac{\frac{5\pi+8\pi}{4}}{3}\right)=\sqrt[6]{32}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)$

$v_3=\sqrt[6]{32}\left(\cos\frac{\frac{5\pi}{4}+4\pi}{3}+i\sin\frac{\frac{5\pi}{4}+\frac{16\pi}{4}}{3}\right)=\sqrt[6]{32}\left(\cos\frac{21\pi}{12}+i\sin\frac{21\pi}{12}\right)$

Napišimo rješenja v_1, v_2 i v_3 u obliku $a+ib$:

$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}$

$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$

Kako je $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

to je $v_1 = \sqrt[6]{32}\left(\frac{\sqrt{6}-\sqrt{2}}{4} + i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$

$\cos\frac{13\pi}{12} = -\cos\frac{\pi}{12} = -\frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin\frac{13\pi}{12} = -\sin\frac{\pi}{12} = -\frac{\sqrt{6}-\sqrt{2}}{4}$

$$v_2 = \sqrt[6]{32}\left(-\frac{\sqrt{6}+\sqrt{2}}{4} - i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$\cos\frac{21\pi}{12} = \cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin\frac{21\pi}{4} = \sin\frac{7\pi}{4} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$$v_3 = \sqrt[6]{32}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$$

v_1, v_2 i v_3 su tražena rješenja

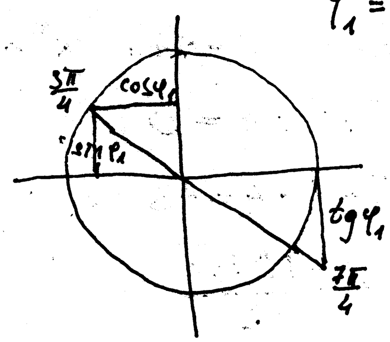
\# Nađi sve vrijednosti korijena $\sqrt[4]{z}$, ako je $z=(-1+i)^8$.

Rj. $\sqrt[4]{z}$, $z = z_1^8$, $z_1 = -1+i$, $|z_1| = \sqrt{2}$

$tg\, 45^\circ = 1$

$\varphi_1 = \frac{3\pi}{4}$

$\cos\varphi_1 = \frac{-1}{\sqrt{2}}$



$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ $\sin\varphi_1 = \frac{1}{\sqrt{2}}$

$z = z_1^8 = \left(\sqrt{2}\right)^8\left[\cos 8\cdot\frac{3\pi}{4} + i\sin 8\cdot\frac{3\pi}{4}\right]$ $tg\,\varphi_1 = \frac{1}{-1} = -1$

$$z = 16(\cos 6\pi + i\sin 6\pi) = 16(\cos 0 + i\sin 0)$$

$\sqrt[4]{z} = ?$ $z_k = \sqrt[4]{|z|}\left(\cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}\right)$

$$z_0 = \sqrt[4]{16}\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) = 2(1 + i\cdot 0) = 2$$

$$z_1 = \sqrt[4]{16}\left(\cos\frac{0+2\pi}{4} + i\sin\frac{0+2\pi}{4}\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i\cdot 1) = 2i$$

$$z_2 = \sqrt[4]{16}\left(\cos\frac{0+4\pi}{4} + i\sin\frac{0+4\pi}{4}\right) = 2(\cos\pi + i\sin\pi) = 2(-1 + i\cdot 0) = -2$$

$$z_3 = \sqrt[4]{16}\left(\cos\frac{0+6\pi}{4} + i\sin\frac{0+6\pi}{4}\right) = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 2(0 + i\cdot(-1)) = -2i$$

Sve vrijednosti $\sqrt[4]{z}$ su $\{2, 2i, -2, -2i\}$

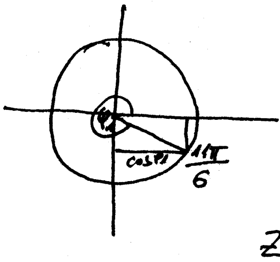
Izračunati $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}\left(2+\sqrt{3}\right)^{12}$.

Rj. Označimo sa $z_1=\sqrt{3}-i$. Tada $|z_1|=\sqrt{3+1}=2$

$\cos\varphi_1=\frac{\sqrt{3}}{2}$ $\left(=\frac{a}{|z|}\right)$

$\sin\varphi_1=-\frac{1}{2}$ $\left(=\frac{b}{|z|}\right)$

$tg\,\varphi_1=\frac{b}{a}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$



$tg\frac{\pi}{6}=30°$.

$\Longrightarrow \varphi_1=\frac{11\pi}{6}=-\frac{\pi}{6}$

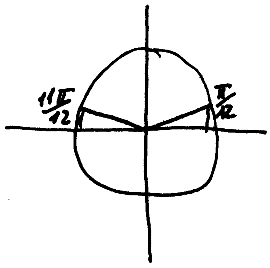
$z_1=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$\left(1-\frac{z_1}{2}\right)=\left(1-\cos\frac{11\pi}{6}-i\sin\frac{11\pi}{6}\right)$ Znamo da je $\cos 2x=\cos^2x-\sin^2x$

$\sin 2x=2\sin x\cos x$

$$\left.\begin{array}{l}1=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\\ \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\end{array}\right\}\Longrightarrow 1-\cos x=2\sin^2\frac{x}{2}$$

$1-\cos\frac{11\pi}{6}=2\sin^2\frac{11\pi}{12}$



$\sin\frac{11\pi}{6}=2\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}$

$$\left(1-\frac{1}{2}z_1\right)=\left(1-\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\left(2\sin^2\frac{11\pi}{12}-2i\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}\right)=$$

$$=2\sin\frac{11\pi}{12}\left(\sin\frac{11\pi}{12}-i\cos\frac{11\pi}{12}\right)=2i\sin\frac{11\pi}{12}\left(-\cos\frac{11\pi}{12}-i\sin\frac{11\pi}{12}\right)=$$

$$=-2i\sin\frac{11\pi}{12}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$

$$\sin\frac{11\pi}{12}=\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{2}(\sqrt{3}-1)}{4},\quad (\sqrt{3}-1)^2=3-2\sqrt{3}+1=4-2\sqrt{3}=2(2-\sqrt{3})$$

$$\sin^2\frac{11\pi}{12}=\sin^2\frac{\pi}{12}=\frac{2(\sqrt{3}-1)^2}{16}=\frac{2(2-\sqrt{3})}{8}=\frac{2-\sqrt{3}}{4},\quad i^{24}=(i^2)^{12}=(-1)^{12}=1$$

$$\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}\left(2+\sqrt{3}\right)^{12}=(-2i)^{24}\left(\sin\frac{11\pi}{12}\right)^{24}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)^{24}\cdot\left(2+\sqrt{3}\right)^{12}$$

$$=(-2)^{24}\left(\sin^2\frac{11\pi}{12}\right)^{12}\left(\cos 24\cdot\frac{11\pi}{12}+i\sin 24\cdot\frac{11\pi}{12}\right)\cdot\left(2+\sqrt{3}\right)^{12}=\cancel{2^{24}}\cdot\frac{(2-\sqrt{3})^{12}}{\cancel{2^{24}}}\cdot$$

$$\cdot(\cos 22\pi+i\sin 22\pi)\cdot(2+\sqrt{3})^{12}=(4-3)^{12}\cdot 1=1$$ traženo rješenje

#) Riješiti jednačinu u skupu kompleksnih brojeva:

$$(2+5i)z^3-2i+5=0.$$

Rj.

$$(2+5i)z^3-2i+5=0$$

$$(2+5i)z^3=2i-5$$

$$z^3=\frac{(2i-5)\cdot(2-5i)}{(2+5i)\cdot(2-5i)}=\frac{4i-10i^2-10+25i}{4-25i^2}=\frac{29i}{29}$$

$$z^3=i$$

$$z=\sqrt[3]{i}$$

Jednačina $z^n=w$ gdje je w kompleksan broj ima n rješenja koje tražimo u obliku

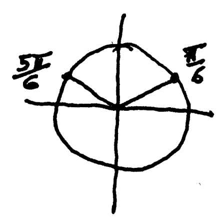
$$z_k=\sqrt[n]{|w|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$$

U našem slučaju $w=i$, $w=a+bi$ $k=0,1,\dots,k-1$

$$|w|=\sqrt{a^2+b^2}=\sqrt{1}=1$$

$$\cos\varphi=\frac{a}{|z|}=0,\quad \sin\varphi=\frac{b}{|z|}=\frac{1}{1}=1 \quad\Rightarrow\quad \varphi=\frac{\pi}{2}$$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$



$$z_0=1\cdot\left(\cos\frac{\frac{\pi}{2}}{3}+i\sin\frac{\pi}{6}\right)=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_1=1\left(\cos\frac{\frac{\pi}{2}+2\pi}{3}+i\sin\frac{\frac{\pi}{2}+2\pi}{3}\right)=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_2=1\left(\cos\frac{\frac{\pi}{2}+4\pi}{3}+i\sin\frac{\frac{\pi}{2}+4\pi}{3}\right)=\cos\frac{9\pi}{6}+i\sin\frac{3\pi}{2}=-i$$

Rješenja jednačine u skupu kompleksnih brojeva su $z_0=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z_1=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ i $z_2=-i$.

(#) Dokazati da je proizvod svih n-tih korijena iz 1 jednak $(-1)^{n-1}$ $(1 \in \mathbb{C})$.

$z = a + ib$

$z = |z|(\cos\varphi + i\sin\varphi)$

Rj. $1 = \cos 0^\circ + i\sin 0^\circ$, $\cos 0^\circ = 1$, $\sin 0^\circ = 0$

$z = 1$, $|z| = \sqrt{a^2+b^2} = 1$, $\varphi = 0$

$\sqrt[n]{1}$ ima n rješenja

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad k = 0,1,2,\dots,n-1$$

u našem slučaju $|z| = 1$, $\varphi = 0$ pa imamo

$$z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0,1,2,\dots,n-1$$

Kako množimo dva kompleksna broja.

$z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$

$z_1 \cdot z_2 = |z_1||z_2|\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right)$

U našem slučaju

$$z_0 \cdot z_1 \cdot z_2 \cdot \ldots \cdot z_{n-1} = (\cos 0^\circ + i\sin 0^\circ)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)\left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right)\cdot$$

$$\ldots\cdot\left(\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right) =$$

$$= \cos\frac{1}{n}\left(2\pi + 4\pi + \ldots + 2(n-1)\pi\right) + i\sin\frac{1}{n}\left(2\pi+4\pi+\ldots+2(n-1)\pi\right) \overset{(*)}{=}$$

Kako sabrati $2 + 4 + 6 + \ldots + 2(n-1)$?

$$S = 2 + 4 + 6 + \ldots + 2(n-1)$$

$$S = 2(n-1) + 2(n-2) + 2(n-3) + \ldots + 2$$

$$2S = \underbrace{2(n-1)+2}_{2n} + \underbrace{2(n-2)+4}_{2n} + \underbrace{2(n-3)+6}_{2n} + \ldots + \underbrace{2(n-1)+2}_{2n}$$

$$2S = (n-1)\cdot 2n \Rightarrow S = (n-1)\cdot n$$

$$\overset{(*)}{=} \cos\frac{1}{n}\cdot(n-1)\cdot n\cdot\pi + i\sin\frac{1}{n}(n-1)\cdot n\cdot\pi = \cos(n-1)\pi + \overbrace{i\sin(n-1)\pi}^{=0 \ \forall n}$$

$= (-1)^{n-1}$ što je i trebalo dobiti

Izračunati $(\sqrt{3}-i)^{2002}$. Rezultat predstaviti u algebarskom obliku.

$z=|z|(\cos\theta+i\sin\theta)$

Rj.

$$\sin 30^\circ=\sin\frac{\pi}{6}=\frac{1}{2}$$

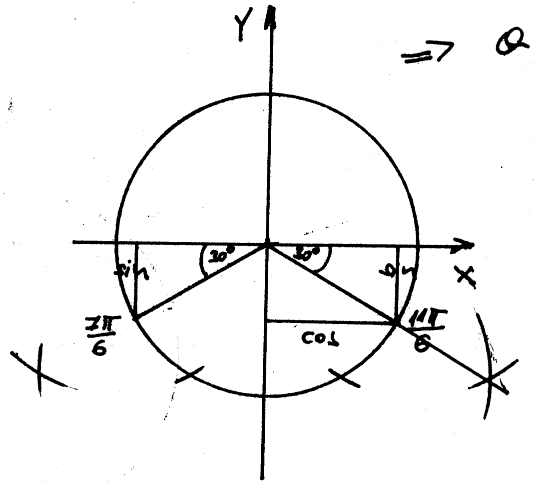
$$z=\sqrt{3}-i$$

$$|z|=\sqrt{3+1}=\sqrt{4}=2$$

$$\cos\theta=\frac{a}{|z|}=\frac{\sqrt{3}}{|z|}=\frac{\sqrt{3}}{2}$$

$$\sin\theta=\frac{b}{|z|}=\frac{-1}{2}$$

$$tg\,\theta=\frac{b}{a}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$$



$$\Rightarrow \theta=\frac{11\pi}{6}$$

$$z=\sqrt{3}-i=$$
$$=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

$$z^n=|z|^n(\cos n\cdot\theta+i\sin n\cdot\theta)$$

$$z=2^{2002}\left(\cos\left(\cancel{2002}^{1001}\cdot\frac{11\pi}{\cancel{6}_{3}}\right)+i\sin\left(\cancel{2002}^{1001}\cdot\frac{11\pi}{\cancel{6}_{3}}\right)\right)=$$

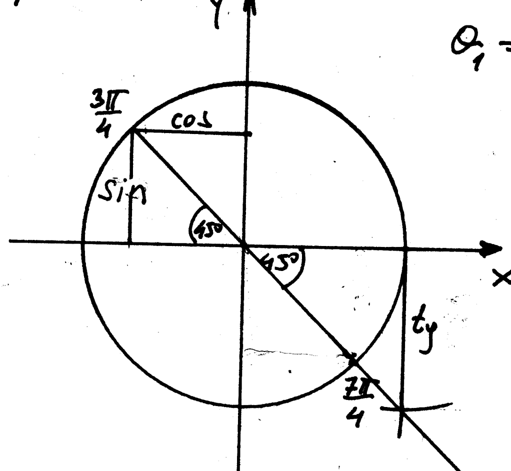
$$=2^{2002}\left(\cos\frac{11\,011}{3}\pi+i\sin\frac{11\,011}{3}\pi\right)=2^{2002}\left(\cos\left(3670\pi+\frac{\pi}{3}\right)\right.$$
$$\left.i\sin\left(3670\pi+\frac{\pi}{3}\right)\right)=2^{2002}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2^{2002}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$z^{2002}=2^{2001}(1+i\sqrt{3})$$

$$(\sqrt{3}-i)^{2002}=2^{2001}(1+i\sqrt{3})$$

Kompleksan broj $z=\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$ napisati u trigonometriskom obliku.

uputa:



$$z_1=i-1=-1+i$$

$$\theta_1=\frac{3\pi}{4}$$

$$z_1=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$z=\frac{\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\ \dots$$

$$=\sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$